

LAN property for a linear model with jumps

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Abstract

In this paper, we consider a linear model with jumps driven by a Brownian motion and a compensated Poisson process, whose drift and diffusion coefficients as well as its intensity are unknown parameters. Supposing that the process is observed discretely at high frequency we derive the local asymptotic normality (LAN) property. In order to obtain this result, Malliavin calculus and Girsanov's theorem are applied in order to write the log-likelihood ratio in terms of sums of conditional expectations, for which a central limit theorem for triangular arrays can be applied.

Résumé

La propriété LAN pour un modèle linéaire avec sauts. Dans cet article, nous considérons un modèle linéaire avec sauts dirigé par un mouvement Brownien et un processus de Poisson compensé dont les coefficients et l'intensité dépendent de paramètres inconnus. Supposant que le processus est observé à haute fréquence, nous obtenons la propriété de normalité asymptotique locale. Pour cela, le calcul de Malliavin et le théorème de Girsanov sont appliqués afin d'écrire le logarithme du rapport de vraisemblances comme une somme d'espérances conditionnelles, pour laquelle un théorème centrale limite pour des suites triangulaires peut être appliqué.

1. Introduction and main result

On a complete probability space (Ω, \mathcal{F}, P) , we consider the following stochastic process $X^{\theta, \sigma, \lambda} = (X_t^{\theta, \sigma, \lambda})_{t \geq 0}$ in \mathbb{R} defined by

$$X_t^{\theta, \sigma, \lambda} = x + \theta t + \sigma B_t + N_t - \lambda t, \quad (1)$$

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where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$ independent of B , and we denote by $(\tilde{N}_t^\lambda)_{t \geq 0}$ the compensated Poisson process $\tilde{N}_t^\lambda := N_t - \lambda t$. The parameters $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ are unknown and Θ, Σ and Λ are closed intervals of $\mathbb{R}, \mathbb{R}_+^*$ and \mathbb{R}_+^* , where $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ denote the natural filtration generated by B and N . We denote by $\mathbb{P}_x^{\theta, \sigma, \lambda}$ the probability law induced by the \mathcal{F} -adapted càdlàg stochastic process $X^{\theta, \sigma, \lambda}$ starting at x , and by $\mathbb{E}_x^{\theta, \sigma, \lambda}$ the expectation with respect to $\mathbb{P}_x^{\theta, \sigma, \lambda}$. Let $\xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}}$ and $\xrightarrow{\mathcal{L}(\mathbb{P}_x^{\theta, \sigma, \lambda})}$ denote the convergence in $\mathbb{P}_x^{\theta, \sigma, \lambda}$ -probability and in $\mathbb{P}_x^{\theta, \sigma, \lambda}$ -law, respectively.

For $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$, we consider an equidistant discrete observation of the process $X^{\theta, \sigma, \lambda}$ which is denoted by $X^n = (X_{t_0}, X_{t_1}, \dots, X_{t_n})$, where $t_k = k\Delta_n$ for $k \in \{0, \dots, n\}$, and $\Delta_n \leq 1$. We assume that the high-frequency observation condition holds. That is,

$$n\Delta_n \rightarrow \infty, \quad \text{and} \quad \Delta_n \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2)$$

Let $p(\cdot; (\theta, \sigma, \lambda))$ denote the density of the random vector X^n under the parameter $(\theta, \sigma, \lambda)$. For $(u, v, w) \in \mathbb{R}^3$, set $\theta_n := \theta + \frac{u}{\sqrt{n\Delta_n}}$, $\sigma_n := \sigma + \frac{v}{\sqrt{n}}$, $\lambda_n := \lambda + \frac{w}{\sqrt{n\Delta_n}}$.

The aim of this paper is to prove the following LAN property.

Theorem 1.1 *Assume condition (2). Then, the LAN property holds for all $(\theta, \sigma, \lambda) \in \Theta \times \Sigma \times \Lambda$ with rate of convergence $(\sqrt{n\Delta_n}, \sqrt{n}, \sqrt{n\Delta_n})$ and asymptotic Fisher information matrix $\Gamma(\theta, \sigma, \lambda)$. That is, for all $z = (u, v, w) \in \mathbb{R}^3$, as $n \rightarrow \infty$,*

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta, \sigma, \lambda))} \xrightarrow{\mathcal{L}(\mathbb{P}_x^{\theta, \sigma, \lambda})} z^\top \mathcal{N}(0, \Gamma(\theta, \sigma, \lambda)) - \frac{1}{2} z^\top \Gamma(\theta, \sigma, \lambda) z,$$

where $\mathcal{N}(0, \Gamma(\theta, \sigma, \lambda))$ is a centered \mathbb{R}^3 -valued Gaussian vector with covariance matrix

$$\Gamma(\theta, \sigma, \lambda) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 + \frac{\sigma^2}{\lambda} \end{pmatrix}.$$

Theorem 1.1 extends in the linear case and in the presence of jumps the results of Gobet in [4] and [5] for multidimensional continuous elliptic diffusions. The main idea of these papers is to use the Malliavin calculus in order to obtain an expression for the derivative of the log-likelihood function in terms of a conditional expectation. Some extensions of Gobet's work with the presence of jumps are given for e.g. in [1], [3], and [7]. However, in the present note, we estimate the coefficients and jump intensity parameters at the same time. The main motivation for this paper is to show some of the important properties and arguments in order to prove the LAMN property in the non-linear case whose proof is non-trivial. In particular, we present four important Lemmas of independent interest which will be key elements in dealing with the non-linear case. The key argument consists in conditioning on the number of jumps within the conditional expectation which expresses the transition density and outside it. When these two conditionings relate to different jumps one may use a large deviation principle in the estimate. When they are equal one uses the complementary set. Within all these arguments the Gaussian type upper and lower bounds of the density conditioned on the jumps is again strongly used. This idea seems to have many other uses in the set-up of stochastic differential equations driven by a Brownian motion and a jump process. We remark here that a plain Itô-Taylor expansion would not solve the problem as higher moments of the Poisson process do not become smaller as the expansion order increases.

2. Preliminaries

In this Section we introduce the preliminary results needed for the proof of Theorem 1.1. In order to deal with the likelihood ratio in Theorem 1.1, we will use the following decomposition

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta, \sigma, \lambda))} = \log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta_n, \sigma, \lambda_n))} + \log \frac{p(X^n; (\theta_n, \sigma, \lambda_n))}{p(X^n; (\theta_n, \sigma, \lambda))} + \log \frac{p(X^n; (\theta_n, \sigma, \lambda))}{p(X^n; (\theta, \sigma, \lambda))}. \quad (3)$$

For each of the above terms we will use a mean value theorem and then analyze each term. We start as in Gobet [4] applying the integration by parts formula of the Malliavin calculus on each interval $[t_k, t_{k+1}]$ to obtain the following expressions for the derivatives of the log-likelihood function w.r.t. θ and σ . Moreover, using Girsanov's theorem, we obtain the following expression for the log-likelihood function w.r.t. λ . For any $t > s$, we denote by $p^{\theta, \sigma, \lambda}(t - s, x, y)$ the transition density of $X_t^{\theta, \sigma, \lambda}$ conditioned on $X_s^{\theta, \sigma, \lambda} = x$.

Proposition 2.1 *For all $\theta \in \mathbb{R}, \sigma, \lambda \in \mathbb{R}_+^*$, and $k \in \{0, \dots, n-1\}$,*

$$\begin{aligned} \frac{\partial_\theta p^{\theta, \sigma, \lambda}}{p^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) &= \frac{1}{\sigma} \mathbb{E}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[B_{t_{k+1}} - B_{t_k} \middle| X_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right], \\ \frac{\partial_\sigma p^{\theta, \sigma, \lambda}}{p^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) &= \frac{1}{\Delta_n} \mathbb{E}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[(B_{t_{k+1}} - B_{t_k})^2 \middle| X_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right] - \frac{1}{\sigma}, \\ \frac{\partial_\lambda p^{\theta, \sigma, \lambda}}{p^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) &= \mathbb{E}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[-\frac{B_{t_{k+1}} - B_{t_k}}{\sigma} + \frac{\tilde{N}_{t_{k+1}}^\lambda - \tilde{N}_{t_k}^\lambda}{\lambda} \middle| X_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right]. \end{aligned}$$

We next present the four Lemmas mentioned in the Introduction. Consider the events $J_m := \{N_{t_{k+1}} - N_{t_k} = m\}$, for all $m \geq 0$ and $k \in \{0, \dots, n-1\}$.

Lemma 2.1 *For all $\theta \in \mathbb{R}, \sigma, \lambda \in \mathbb{R}_+^*$, $k \in \{0, \dots, n-1\}$, and $m \geq 0$,*

$$\mathbb{P}_{X_{t_k}}^{\theta, \sigma, \lambda} \left(J_m \middle| X_{t_{k+1}}^{\theta, \sigma, \lambda} = X_{t_{k+1}} \right) = \frac{e^{-(X_{t_{k+1}} - X_{t_k} - m - (\theta - \lambda)\Delta_n)^2 / (2\sigma^2 \Delta_n)} \frac{(\lambda \Delta_n)^m}{m!}}{\sum_{i=0}^{\infty} e^{-(X_{t_{k+1}} - X_{t_k} - i - (\theta - \lambda)\Delta_n)^2 / (2\sigma^2 \Delta_n)} \frac{(\lambda \Delta_n)^i}{i!}}.$$

For all $j, p \geq 0$ and $k \in \{0, \dots, n-1\}$, we introduce the random variable

$$S_j^p := \mathbf{1}_{J_j} \mathbb{E}_{X_{t_k}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} \left[\mathbf{1}_{J_j^c} (N_{t_{k+1}} - N_{t_k})^p \middle| X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = X_{t_{k+1}} \right].$$

We remark that heuristically the indicator functions $\mathbf{1}_{J_j}$ and $\mathbf{1}_{J_j^c}$ outside and inside the conditional expectation correspond to restrictions on $X^{(\theta, \sigma, \lambda)}$ and $X^{(\bar{\theta}, \bar{\sigma}, \bar{\lambda})}$, respectively.

Lemma 2.2 *For all $\theta, \bar{\theta} \in \mathbb{R}, \sigma, \bar{\sigma}, \lambda, \bar{\lambda} \in \mathbb{R}_+^*$, $j, p \geq 0$ and $k \in \{0, \dots, n-1\}$,*

$$S_j^p = \mathbf{1}_{J_j} \frac{\sum_{m=0: m \neq j}^{\infty} m^p e^{-(\sigma(B_{t_{k+1}} - B_{t_k}) + j - m + (\theta - \bar{\theta} - \lambda + \bar{\lambda})\Delta_n)^2 / (2\sigma^2 \Delta_n)} \frac{(\bar{\lambda} \Delta_n)^m}{m!}}{\sum_{i=0}^{\infty} e^{-(\sigma(B_{t_{k+1}} - B_{t_k}) + j - i + (\theta - \bar{\theta} - \lambda + \bar{\lambda})\Delta_n)^2 / (2\bar{\sigma}^2 \Delta_n)} \frac{(\bar{\lambda} \Delta_n)^i}{i!}}. \quad (4)$$

We next fix $\alpha \in (0, \frac{1}{2})$, and analyze S_j^p in two separate cases as follows

$$S_j^p = S_j^p \mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| \leq \Delta_n^\alpha\}} + S_j^p \mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| > \Delta_n^\alpha\}} =: S_{1,j}^p + S_{2,j}^p.$$

Furthermore, we write $S_{1,j}^p = S_{1,1,j}^p + S_{1,2,j}^p$, and $S_{2,j}^p = S_{2,1,j}^p + S_{2,2,j}^p$, where $S_{1,1,j}^p$ and $S_{2,1,j}^p$ contain the terms $\sum_{m < j}$, and $S_{1,2,j}^p$ and $S_{2,2,j}^p$ contain the terms $\sum_{m > j}$ in (4).

Lemma 2.3 *Assume that $|\theta - \bar{\theta}| \leq \frac{C}{\sqrt{n\Delta_n}}$ and $|\lambda - \bar{\lambda}| \leq \frac{C}{\sqrt{n\Delta_n}}$, for some constant $C > 0$. Then for all $\sigma, \bar{\sigma} \in \mathbb{R}_+^*, j, p \geq 0$, $k \in \{0, \dots, n-1\}$, and for n large enough,*

$$S_{1,1,j}^p \leq \mathbf{1}_{J_j} \frac{j!}{(\bar{\lambda}\Delta_n)^j} \sum_{m < j} m^p e^{-\frac{(j-m)^2}{4\bar{\sigma}^2\Delta_n}} \frac{(\bar{\lambda}\Delta_n)^m}{m!}, \quad S_{1,2,j}^p \leq \mathbf{1}_{J_j} e^{-\frac{1}{4\bar{\sigma}^2\Delta_n}} \sum_{\ell > 0} (\ell + j)^p \frac{(\bar{\lambda}\Delta_n)^\ell}{\ell!},$$

$$S_{2,1,j}^p \leq j^p \mathbf{1}_{J_j} \mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| > \Delta_n^\alpha\}}, \quad S_{2,2,j}^p \leq \mathbf{1}_{J_j} \mathbf{1}_{\{|B_{t_{k+1}} - B_{t_k}| > \Delta_n^\alpha\}} \sum_{\ell=0}^{\infty} (\ell + j + 1)^p \frac{(\bar{\lambda}\Delta_n)^\ell}{\ell!}.$$

For all $p \geq 0$ and $k \in \{0, \dots, n-1\}$, set

$$M_{1,p}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} := \sum_{j=0}^{\infty} j^p \mathbb{E}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{J_j} \mathbb{E}_{X_{t_k}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} \left[\mathbf{1}_{J_j^c} \left| X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = X_{t_{k+1}} \right| \right] \right],$$

$$M_{2,p}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} := \sum_{j=0}^{\infty} \mathbb{E}_{X_{t_k}}^{\theta, \sigma, \lambda} \left[\mathbf{1}_{J_j} \mathbb{E}_{X_{t_k}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} \left[\mathbf{1}_{J_j^c} (N_{t_{k+1}} - N_{t_k})^p \left| X_{t_{k+1}}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} = X_{t_{k+1}} \right| \right] \right].$$

Lemma 2.4 Assume that $|\theta - \bar{\theta}| \leq \frac{C}{\sqrt{n\Delta_n}}$ and $|\lambda - \bar{\lambda}| \leq \frac{C}{\sqrt{n\Delta_n}}$, for some constant $C > 0$. Then, for any $\sigma, \bar{\sigma} \in \Sigma$, $p \geq 0$, and for n large enough, there exist constants $C_1, C_2 > 0$ such that for all $\alpha \in (0, \frac{1}{2})$, and $k \in \{0, \dots, n-1\}$,

$$M_{1,p}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} + M_{2,p}^{\bar{\theta}, \bar{\sigma}, \bar{\lambda}} \leq C_1 e^{-\frac{1}{C_2 \Delta_n^{1-2\alpha}}}.$$

We next recall a convergence in probability result, and a central limit theorem for triangular arrays of random variables. For each $n \in \mathbb{N}$, let $(Z_{k,n})_{k \geq 1}$ and $(\zeta_{k,n})_{k \geq 1}$ be two sequences of random variables defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and assume that they are $\mathcal{F}_{t_{k+1}}$ -measurable.

Lemma 2.5 [2, Lemma 9] Assume that $\sum_{k=0}^{n-1} \mathbb{E}[Z_{k,n} | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0$, and $\sum_{k=0}^{n-1} \mathbb{E}[Z_{k,n}^2 | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$. Then $\sum_{k=0}^{n-1} Z_{k,n} \xrightarrow{\mathbb{P}} 0$, as $n \rightarrow \infty$.

Lemma 2.6 [6, Lemma 4.3] Assume that there exist real numbers M and $V > 0$ such that as $n \rightarrow \infty$,

$$\sum_{k=0}^{n-1} \mathbb{E}[\zeta_{k,n} | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} M, \quad \sum_{k=0}^{n-1} \left(\mathbb{E}[\zeta_{k,n}^2 | \mathcal{F}_{t_k}] - (\mathbb{E}[\zeta_{k,n} | \mathcal{F}_{t_k}])^2 \right) \xrightarrow{\mathbb{P}} V, \quad \text{and} \quad \sum_{k=0}^{n-1} \mathbb{E}[\zeta_{k,n}^4 | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}} 0.$$

Then as $n \rightarrow \infty$, $\sum_{k=0}^{n-1} \zeta_{k,n} \xrightarrow{\mathcal{L}(\mathbb{P})} \mathcal{N} + M$, where \mathcal{N} is a centered Gaussian variable with variance V .

3. Proof of Theorem 1.1

For $\ell \in [0, 1]$, set $\theta(\ell) := \theta_n(\ell, u) := \theta + \frac{\ell u}{\sqrt{n\Delta_n}}$, $\sigma(\ell) := \sigma_n(\ell, v) := \sigma + \frac{\ell v}{\sqrt{n}}$, $\lambda(\ell) := \lambda_n(\ell, w) := \lambda + \frac{\ell w}{\sqrt{n\Delta_n}}$. Applying the Markov property and Proposition 2.1 to each term in (3), we obtain that

$$\begin{aligned} \log \frac{p(X^n; (\theta_n, \sigma, \lambda))}{p(X^n; (\theta, \sigma, \lambda))} &= \sum_{k=0}^{n-1} \log \frac{p^{\theta_n, \sigma, \lambda}}{p^{\theta, \sigma, \lambda}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) \\ &= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n}} \int_0^1 \frac{\partial_\theta p^{\theta(\ell), \sigma, \lambda}}{p^{\theta(\ell), \sigma, \lambda}}(\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell \\ &= \sum_{k=0}^{n-1} \frac{u}{\sqrt{n\Delta_n}} \frac{1}{\sigma} \int_0^1 \mathbb{E}_{X_{t_k}}^{\theta(\ell), \sigma, \lambda} \left[B_{t_{k+1}} - B_{t_k} \left| X_{t_{k+1}}^{\theta(\ell), \sigma, \lambda} = X_{t_{k+1}} \right| \right] d\ell, \end{aligned}$$

$$\begin{aligned} \log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta_n, \sigma, \lambda_n))} &= \sum_{k=0}^{n-1} \frac{v}{\sqrt{n}} \int_0^1 \frac{\partial_\sigma p^{\theta_n, \sigma(\ell), \lambda_n}}{p^{\theta_n, \sigma(\ell), \lambda_n}} (\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell \\ &= \sum_{k=0}^{n-1} \frac{v}{\sqrt{n}} \int_0^1 \left(\frac{1}{\Delta_n} \mathbb{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[(B_{t_{k+1}} - B_{t_k})^2 \middle| X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] - \frac{1}{\sigma(\ell)} \right) d\ell, \end{aligned}$$

and

$$\begin{aligned} \log \frac{p(X^n; (\theta_n, \sigma, \lambda_n))}{p(X^n; (\theta_n, \sigma, \lambda))} &= \sum_{k=0}^{n-1} \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \frac{\partial_\lambda p^{\theta_n, \sigma, \lambda(\ell)}}{p^{\theta_n, \sigma, \lambda(\ell)}} (\Delta_n, X_{t_k}, X_{t_{k+1}}) d\ell \\ &= \sum_{k=0}^{n-1} \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \mathbb{E}_{X_{t_k}}^{\theta_n, \sigma, \lambda(\ell)} \left[-\frac{B_{t_{k+1}} - B_{t_k}}{\sigma} + \frac{\tilde{N}_{t_{k+1}}^{\lambda(\ell)} - \tilde{N}_{t_k}^{\lambda(\ell)}}{\lambda(\ell)} \middle| X_{t_{k+1}}^{\theta_n, \sigma, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell. \end{aligned}$$

Now using equation (1), we obtain the following expansion of the log-likelihood ratio

$$\log \frac{p(X^n; (\theta_n, \sigma_n, \lambda_n))}{p(X^n; (\theta, \sigma, \lambda))} = \sum_{k=0}^{n-1} (\xi_{k,n} + H_{k,n} + \eta_{k,n} + M_{k,n} + \beta_{k,n} - R_{k,n}),$$

where

$$\begin{aligned} \xi_{k,n} &:= \frac{u}{\sqrt{n\Delta_n}} \frac{1}{\sigma^2} \left(\sigma (B_{t_{k+1}} - B_{t_k}) - \frac{u\Delta_n}{2\sqrt{n\Delta_n}} \right), \\ H_{k,n} &:= \frac{u}{\sqrt{n\Delta_n}} \frac{1}{\sigma^2} \left(\tilde{N}_{t_{k+1}}^\lambda - \tilde{N}_{t_k}^\lambda - \int_0^1 \mathbb{E}_{X_{t_k}}^{\theta(\ell), \sigma, \lambda} \left[\tilde{N}_{t_{k+1}}^\lambda - \tilde{N}_{t_k}^\lambda \middle| X_{t_{k+1}}^{\theta(\ell), \sigma, \lambda} = X_{t_{k+1}} \right] d\ell \right), \\ \eta_{k,n} &:= \frac{v}{\sqrt{n}} \int_0^1 \frac{1}{\Delta_n} \left(\frac{\sigma^2}{\sigma(\ell)^3} (B_{t_{k+1}} - B_{t_k})^2 - \frac{\Delta_n}{\sigma(\ell)} \right) d\ell, \\ M_{k,n} &:= \frac{v}{\sqrt{n}} \int_0^1 \frac{1}{\Delta_n} \frac{1}{\sigma(\ell)^3} \left\{ \left(\theta\Delta_n + \tilde{N}_{t_{k+1}}^\lambda - \tilde{N}_{t_k}^\lambda \right)^2 + 2\sigma (B_{t_{k+1}} - B_{t_k}) \left(\theta\Delta_n + \tilde{N}_{t_{k+1}}^\lambda - \tilde{N}_{t_k}^\lambda \right) \right. \\ &\quad \left. - \mathbb{E}_{X_{t_k}}^{\theta_n, \sigma(\ell), \lambda_n} \left[\left(\theta_n\Delta_n + \tilde{N}_{t_{k+1}}^{\lambda_n} - \tilde{N}_{t_k}^{\lambda_n} \right)^2 \right. \right. \\ &\quad \left. \left. + 2\sigma(\ell) (B_{t_{k+1}} - B_{t_k}) \left(\theta_n\Delta_n + \tilde{N}_{t_{k+1}}^{\lambda_n} - \tilde{N}_{t_k}^{\lambda_n} \right) \middle| X_{t_{k+1}}^{\theta_n, \sigma(\ell), \lambda_n} = X_{t_{k+1}} \right] \right\} d\ell, \\ \beta_{k,n} &:= -\frac{w}{\sqrt{n\Delta_n}} \frac{1}{\sigma^2} \left(\sigma (B_{t_{k+1}} - B_{t_k}) + \frac{w\Delta_n}{2\sqrt{n\Delta_n}} - \frac{u\Delta_n}{\sqrt{n\Delta_n}} \right) \\ &\quad + \frac{w}{\sqrt{n\Delta_n}} \int_0^1 \mathbb{E}_{X_{t_k}}^{\theta_n, \sigma, \lambda(\ell)} \left[\frac{\tilde{N}_{t_{k+1}}^{\lambda(\ell)} - \tilde{N}_{t_k}^{\lambda(\ell)}}{\lambda(\ell)} \middle| X_{t_{k+1}}^{\theta_n, \sigma, \lambda(\ell)} = X_{t_{k+1}} \right] d\ell, \\ R_{k,n} &:= \frac{w}{\sqrt{n\Delta_n}} \frac{1}{\sigma^2} \int_0^1 \left(\tilde{N}_{t_{k+1}}^{\lambda(\ell)} - \tilde{N}_{t_k}^{\lambda(\ell)} - \mathbb{E}_{X_{t_k}}^{\theta_n, \sigma, \lambda(\ell)} \left[\tilde{N}_{t_{k+1}}^{\lambda(\ell)} - \tilde{N}_{t_k}^{\lambda(\ell)} \middle| X_{t_{k+1}}^{\theta_n, \sigma, \lambda(\ell)} = X_{t_{k+1}} \right] \right) d\ell. \end{aligned}$$

We next show that the random variables $\xi_{k,n}, \eta_{k,n}, \beta_{k,n}$ are the terms that contribute to the limit in Theorem 1.1, and $H_{k,n}, M_{k,n}$ and $R_{k,n}$ are the negligible contributions. Indeed, using Girsanov's theorem and Lemma 2.4, we can show that the conditions of Lemma 2.5 under $\mathbb{P}_x^{\theta, \sigma, \lambda}$ hold for each term $H_{k,n}, M_{k,n}$ and $R_{k,n}$. That is,

Lemma 3.1 *Assume condition (2). Then, as $n \rightarrow \infty$,*

$$\begin{aligned}
& \sum_{k=0}^{n-1} (H_{k,n} + M_{k,n} - R_{k,n}) \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} 0 \\
& \sum_{k=0}^{n-1} \mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} + \eta_{k,n} + \beta_{k,n} | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} -\frac{u^2}{2\sigma^2} - \frac{v^2}{2} \frac{2}{\sigma^2} - \frac{w^2}{2\sigma^2} \left(1 + \frac{\sigma^2}{\lambda}\right) + \frac{uw}{\sigma^2} \\
& \sum_{k=0}^{n-1} \left(\mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n}^2 + \eta_{k,n}^2 + \beta_{k,n}^2 | \mathcal{F}_{t_k}] - \mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} | \mathcal{F}_{t_k}]^2 - \mathbb{E}^{\theta, \sigma, \lambda} [\eta_{k,n} | \mathcal{F}_{t_k}]^2 - \mathbb{E}^{\theta, \sigma, \lambda} [\beta_{k,n} | \mathcal{F}_{t_k}]^2 \right) \\
& \quad \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} \frac{u^2}{\sigma^2} + 2\frac{v^2}{\sigma^2} + \frac{w^2}{\sigma^2} \left(1 + \frac{\sigma^2}{\lambda}\right) \\
& \sum_{k=0}^{n-1} \mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n}^4 + \eta_{k,n}^4 + \beta_{k,n}^4 | \mathcal{F}_{t_k}] \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} 0 \\
& \sum_{k=0}^{n-1} \left(\mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} \eta_{k,n} | \mathcal{F}_{t_k}] - \mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} | \mathcal{F}_{t_k}] \mathbb{E}^{\theta, \sigma, \lambda} [\eta_{k,n} | \mathcal{F}_{t_k}] \right) \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} 0 \\
& \sum_{k=0}^{n-1} \left(\mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} \beta_{k,n} | \mathcal{F}_{t_k}] - \mathbb{E}^{\theta, \sigma, \lambda} [\xi_{k,n} | \mathcal{F}_{t_k}] \mathbb{E}^{\theta, \sigma, \lambda} [\beta_{k,n} | \mathcal{F}_{t_k}] \right) \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} -\frac{uw}{\sigma^2} \\
& \sum_{k=0}^{n-1} \left(\mathbb{E}^{\theta, \sigma, \lambda} [\eta_{k,n} \beta_{k,n} | \mathcal{F}_{t_k}] - \mathbb{E}^{\theta, \sigma, \lambda} [\eta_{k,n} | \mathcal{F}_{t_k}] \mathbb{E}^{\theta, \sigma, \lambda} [\beta_{k,n} | \mathcal{F}_{t_k}] \right) \xrightarrow{\mathbb{P}_x^{\theta, \sigma, \lambda}} 0.
\end{aligned}$$

Finally, Lemma 2.6 applied to $\zeta_{k,n} = \xi_{k,n} + \eta_{k,n} + \beta_{k,n}$ concludes the proof of Theorem 1.1.

Acknowledgements

Second author acknowledges support from the European Union programme FP7-PEOPLE-2012-CIG under grant agreement 333938. Third author acknowledges support from the LIA CNRS Formath Vietnam and the program ARCUS MAE/IDF Vietnam.

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